

On necessary and sufficient conditions for the variable exponent Hardy type inequality

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Abstract

We derive a number of equivalent criterions for the variable exponent Hardy type inequality

$$\left\| \frac{1}{x} \int_0^x f(t) dt \right\|_{L^{p(\cdot)}(0,1)} \leq C \|f\|_{L^{p(\cdot)}(0,1)}; f \geq 0.$$

to hold, whenever the exponent $p : (0,1) \rightarrow (1,\infty)$ is increasing or decreasing near small neighborhood of the origin.

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1 Introduction

We study Hardy's inequality

$$\|x^{-1} Hf\|_{L^{p(\cdot)}(0,1)} \leq C \|f\|_{L^{p(\cdot)}(0,1)} \quad (1.1)$$

in the norms of variable exponent Lebesgue space $L^{p(\cdot)}(0,1)$. Here $Hf(x) = \int_0^x f(t) dt$ is Hardy's operator and the constant $C > 0$ does not depend on arbitrary positive measurable function f . This subject has been studied by several authors (see, e.g. [2], [4], [5], [7], [8], [9], [11], [12], [13], [14], [15], [16], [17]).

There are several sufficient conditions on the function $p : (0, 1) \rightarrow (1, \infty)$ for the inequality (1.1) to hold. They are expressed in terms of regularity conditions for p at the origin. It follows from the results of works [4], [9], [15] (see, also [2], [12], [14]) that the inequality (1.1) holds if $p^- = \inf p > 1$, $p^+ = \sup p(x) < \infty$ and the condition

$$A := \limsup_{x \rightarrow 0} |p(x) - p(0)| \log \frac{1}{x} < \infty. \quad (1.2)$$

is satisfied.

One can think that the inequality (1.1) does not need for a condition type of (1.2) at all. Since there exists an example of function p for which the inequality (1.1) is violated by some sequence of functions $\{f_k\}$ (see, [9], [7]), we see that the inequality (1.1) does not hold without restriction on p (Note, the p there is not monotone and does not satisfy (1.2)). In [11] (see, also [7]), we had proved that the condition

$$B := \limsup_{x \rightarrow 0} \left[p(x) - p\left(\frac{x}{2}\right) \right] \log \frac{1}{x} < \infty \quad (1.3)$$

is necessary for this case. Note that, condition (1.3) is strictly weaker than (1.2). This condition is new and somewhat surprising. For example, it is satisfied by $p(x) = p(0) + \frac{C}{(\ln \frac{1}{x})^\alpha}$ and $0 < \alpha < 1$, $C > 0$, whereas the condition (1.2) is not satisfied. For the exponent, that is nondecreasing near the origin, the condition (1.3) is also sufficient if the number B satisfies $B < p(0)(p(0) - 1)$ (see, [11]). Unfortunately, the good condition (1.3) is no longer sufficient for the inequality (1.1) to hold if the condition on B be ignored. In this case, a necessary and sufficient condition is still an open problem.

In Theorem 2.2, we prove that the condition

$$\int_a^1 \left(a^{\frac{1}{p'(a)}} x^{-\frac{1}{p'(x)}} \right)^{p(x)} \frac{dx}{x} \leq C, \quad 0 < a < 1 \quad (1.4)$$

and several other equivalent conditions are necessary and sufficient for the inequality (1.1) to hold in the case of nondecreasing exponents.

Also, in Theorem 2.1, we prove that no condition is needed if the exponent p is nonincreasing at small neighborhood of the origin.

We refer to the monograph [3] and references therein for a full description of variable exponent Lebesgue spaces and boundedness of classical integral operators there.

2 Main results and notation

As to the basic properties of spaces $L^{p(\cdot)}$, we refer to [6], [18]. Throughout this paper, it is assumed that $p(x)$ is a measurable function in $(0, 1)$, taking its values from the interval $[1, \infty)$ with $p^+ = \sup \{p(x) : x \in (0, 1)\} < \infty$. The space of functions $L^{p(\cdot)}(0, 1)$ is introduced as the class of measurable functions

$f(x)$ on $(0, 1)$ which have a finite $I_{p(\cdot)}(f) = \int_0^1 |f|^{p(x)} dx$ modular. A norm in $L^{p(\cdot)}(0, 1)$ is given in the form

$$\|f\| = \left\{ \lambda > 0 : I_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

For $1 < p^-, p^+ < \infty$ the space $L^{p(\cdot)}(0, 1)$ is a reflexive Banach space.

The relation between modular and norm is expressed by the following inequalities (see, f.e. [18]):

$$\|f\|_{L^{p(\cdot)}(0,1)}^{p^+} \leq I_p(f) \leq \|f\|_{L^{p(\cdot)}(0,1)}^{p^-}, \quad 1 \geq \|f\|_{p(\cdot)}, \quad (2.1)$$

$$\|f\|_{L^{p(\cdot)}(0,1)}^{p^-} \leq I_p(f) \leq \|f\|_{L^{p(\cdot)}(0,1)}^{p^+}, \quad 1 \leq \|f\|_{p(\cdot)}. \quad (2.2)$$

Such estimates allow us to perform our estimates in terms of a modular.

For the function $1 \leq p(x) < \infty$ $p'(x)$ denotes the conjugate function of $p(x)$, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and $p' = \infty$ if $p = 1$. We denote by C, C_1, C_2, \dots various positive constants whose values may vary at each appearance. By χ_E we denote the characteristic function of set E . We say the function f is almost increasing (almost decreasing) on $[0, 1]$ if $f(x) \leq Cf(y)$ ($f(y) \leq Cf(x)$) for all $x \leq y$ in $[0, 1]$ and $C > 0$.

Following main results are obtained in this paper.

Theorem 2.1 *Let $p : (0, 1) \rightarrow [1, \infty)$ be a measurable function such that p is nonincreasing on some interval $(0, \epsilon)$, $\epsilon > 0$ and $p^+ < \infty$. Then it holds the inequality (1.1) for any positive measurable function f .*

Theorem 2.2 *Let $p : (0, 1) \rightarrow [1, \infty)$ be a nondecreasing function such that $p(1) < \infty$. Then the following statements are equivalent:*

1. *There exists a constant $C > 0$ such that the inequality*

$$\|x^{-1}Hf\|_{L^{p(\cdot)}(0,1)} \leq C\|f\|_{L^{p(\cdot)}(0,1)} \quad (2.3)$$

holds for any positive measurable function f .

2. *The condition*

$$\int_a^1 x^{-\frac{1}{p'(x)}} \frac{dx}{x} \leq Ca^{-\frac{1}{p'(a)}}, \quad 0 < a < 1 \quad (2.4)$$

is satisfied.

3. *There exists an $\epsilon > 0$ such that the function $x^{-\frac{1}{p'(x)}+\epsilon}$ is almost decreasing:*

$$t_2^{-\frac{1}{p'(t_2)}+\epsilon} \leq Ct_1^{-\frac{1}{p'(t_1)}+\epsilon} \quad \text{as} \quad 0 < t_1 \leq t_2 < 1. \quad (2.5)$$

4. The condition (4.16) is satisfied.

5. The condition

$$\|x^{-1}\|_{p(\cdot);(a,1)} \leq Ca^{-\frac{1}{p'(a)}}, \quad 0 < a < 1. \quad (2.6)$$

is satisfied.

3 Proof of Theorem 2.1.

Let $f(x) \geq 0$ be a measurable function such that $\|f\|_{L^{p(\cdot)}(0,1)} \leq 1$. Then it follows from the inequality (2.1) that $I_{p(\cdot)}(f) \leq 1$. In order to prove Theorem 2.1 we have to show that

$$\|x^{-1}Hf\|_{L^{p(\cdot)}(0,1)} \leq C_1. \quad (3.1)$$

To prove (3.1), we establish the estimate

$$I_{p(\cdot)}\left(\frac{Hf}{x}\right) \leq C_2.$$

Using triangle inequality for $p(\cdot)$ -norms and $\epsilon \in (0, 1)$, we have

$$\begin{aligned} \|x^{-1}Hf\|_{L^{p(\cdot)}(0,1)} &\leq \|x^{-1}Hf\|_{L^{p(\cdot)}(0,\epsilon)} + \|x^{-1}Hf\|_{L^{p(\cdot)}(\epsilon,1)} \\ &:= i_1 + i_2. \end{aligned} \quad (3.2)$$

Taking into account

$$\frac{Hf(x)}{x} = \int_0^1 f(tx)dt$$

and using Minkowskii's inequality for $L^{p(\cdot)}$ norms, it follows that (see, [6], [18])

$$i_1 = \left\| \frac{Hf}{x} \right\|_{p(\cdot);(0,\epsilon)} \leq \left\| \int_0^1 f(\cdot t)dt \right\|_{p(\cdot);(0,\epsilon)} \leq \int_0^1 \|f(\cdot t)\|_{p(\cdot);(0,\epsilon)} dt. \quad (3.3)$$

Let us estimate the term $\|f(\cdot t)\|_{p(\cdot);(0,\epsilon)}$ for $0 < t < 1$. Since p is nonincreasing on $(0, \epsilon)$, we have $p(x) \leq p(tx)$ for $x \in (0, \epsilon)$. Therefore,

$$\begin{aligned} \int_0^\epsilon f(xt)^{p(x)} dx &\leq \int_0^\epsilon f(xt)^{p(x)} \chi_{f(xt) \geq 1} dx + \int_0^\epsilon dx \\ &\leq \epsilon + \int_0^\epsilon f(tx)^{p(tx)} \chi_{f(tx) \geq 1} = \epsilon + \frac{1}{t} \int_0^{t\epsilon} f(u)^{p(u)} du. \end{aligned}$$

Whence,

$$\int_0^\epsilon f(tx)^{p(x)} \leq \frac{1}{t} + \epsilon \leq \frac{2}{t}, \quad 0 < t < 1.$$

This implies

$$\int_0^\epsilon \left(\frac{f(tx)}{t^{-\frac{1}{p^-}} 2^{\frac{1}{p^-}}} \right)^{p(x)} dx \leq 1, \quad 0 < t < 1.$$

Therefore and using the definition of $p(\cdot)$ -norms, we get

$$\|f(\cdot t)\|_{p(\cdot); (0, \epsilon)} \leq 2^{\frac{1}{p^-}} t^{-\frac{1}{p^-}}, \quad 0 < t < 1. \quad (3.4)$$

Using (3.4) and (3.3) for the first summand in (3.2) we have the estimate

$$i_1 \leq 2^{\frac{1}{p^-}} \int_0^1 t^{-\frac{1}{p^-}} dt \leq \frac{p^-}{p^- - 1} 2^{\frac{1}{p^-}}. \quad (3.5)$$

Now we shall estimate the term $\left\| \frac{Hf(\cdot)}{\cdot} \right\|_{p(\cdot); (\epsilon, 1)}$. For $x \in (\epsilon, 1)$ using Young's inequality, we get

$$\begin{aligned} \int_0^1 f(tx) dt &\leq \int_0^1 \frac{f(tx)^{p(tx)}}{p(tx)} dt + \int_0^1 \frac{dt}{p'(tx)} \\ \frac{1}{xp^-} \int_0^x f(u)^{p(u)} du + \frac{p^- - 1}{p^-} &\leq \frac{1}{\epsilon p^-} + \frac{1}{(p^+)' } \leq 1 + \frac{1}{\epsilon}. \end{aligned}$$

Therefore,

$$\begin{aligned} i_2 &= \left\| \frac{Hf(\cdot)}{\cdot} \right\|_{p(\cdot); (\epsilon, 1)} = \left\| \int_0^1 f(\cdot t) dt \right\|_{p(\cdot); (\epsilon, 1)} \\ &\leq \left(\frac{1}{\epsilon} + 1 \right) \|1\|_{p(\cdot); (\epsilon, 1)} \leq C. \end{aligned}$$

Inserting this estimate and (3.5) in (3.2) we complete the proof of Theorem 2.1.

4 Proof of Theorem 2.2.

To prove Theorem 2.2 we need several lemmas.

Lemma 4.1 *Let $p : (0, 1) \rightarrow [1, \infty)$ be a monotone nondecreasing function such that $p(1) < \infty$ and the condition (4.16) is satisfied. Then there exists a constant $C_1 > 0$ depending on $C, p(1)$ such that the condition*

$$\left| \frac{1}{p'(2x)} - \frac{1}{p'(x)} \right| \ln \frac{1}{x} \leq C_1 \quad (4.1)$$

is satisfied.

Proof. From (4.16) it follows that

$$\int_{2a}^{4a} \left(x^{-\frac{1}{p'(x)}} a^{\frac{1}{p'(a)}} \right)^{p(x)} \frac{dx}{x} \leq C.$$

Since $\frac{1}{p'(x)}$ is monotone nondecreasing, we have

$$\int_{2a}^{4a} \left((4a)^{-\frac{1}{p'(2a)}} a^{\frac{1}{p'(a)}} \right)^{p(x)} \frac{dx}{x} \leq C.$$

Suppose $a^{\frac{1}{p'(a)}} (4a)^{-\frac{1}{p'(2a)}}$ is greeter then 1. Then

$$C \geq \left((4a)^{-\frac{1}{p'(2a)}} a^{\frac{1}{p'(a)}} \right)^{p(0)} \ln 2 \geq 4^{1-p(0)} \ln 2 a^{\frac{1}{p'(a)} - \frac{1}{p'(2a)}}.$$

Whence,

$$\left(\frac{1}{a} \right)^{\frac{1}{p'(2a)} - \frac{1}{p'(a)}} \leq 1 + \frac{C4^{p(1)-1}}{\ln 2}$$

or

$$\left(\frac{1}{p'(2a)} - \frac{1}{p'(a)} \right) \ln \frac{1}{a} \leq \ln \left(\frac{C4^{p(1)-1}}{\ln 2} + 1 \right)$$

This completes the proof of Lemma 4.1 with constant $C_1 = \ln \left(\frac{C4^{p(1)-1}}{\ln 2} + 1 \right)$. ■

Lemma 4.2 *Let $p : (0, 1) \rightarrow [1, \infty)$ be a nondecreasing function satisfying the condition (4.16) and $p(1) < \infty$. Then there exists a constant $C_1 > 0$ depending on C and $p(0)$ such that for any $\frac{x}{2} \leq y \leq 2x$, $0 < x < \frac{1}{4}$ the estimate*

$$\frac{1}{C_1} \phi(x) \leq \phi(y) \leq C_1 \phi(x) \tag{4.2}$$

holds, where the function $\phi(t) = t^{-\frac{1}{p'(t)}}$.

Proof. Since $\frac{1}{p'}$ is nondecreasing it follows from Lemma 4.1 that

$$\begin{aligned} \phi(y) &\leq \left(\frac{x}{2} \right)^{-\frac{1}{p'(y)}} \leq \left(\frac{1}{x} \right)^{\frac{1}{p'(2x)} - \frac{1}{p'(x)}} x^{-\frac{1}{p'(x)}} 2^{\frac{1}{p'(1)}} \\ &\leq 2 \left(C4^{p(1)-1} + 1 \right) \phi(x). \end{aligned}$$

By the same way,

$$\begin{aligned} \phi(x) &\leq \left(\frac{y}{2} \right)^{-\frac{1}{p'(x)}} \leq \left(\frac{1}{y} \right)^{\frac{1}{p'(2y)} - \frac{1}{p'(y)}} y^{-\frac{1}{p'(y)}} 2^{\frac{1}{p'(1)}} \\ &\leq 2 \left(C4^{p(1)-1} + 1 \right) \phi(y). \end{aligned}$$

Therefore, (4.2) is satisfied by the constant $2 \left(C4^{p(1)-1} + 1 \right)$. ■

Lemma 4.3 *Let $p : (0, 1) \rightarrow [1, \infty)$ be a nondecreasing function such that $p(1) < \infty$ and the condition (4.14) is satisfied. Then there exists a constant $C_1 > 0$ depending on C such that the condition (4.1) is satisfied.*

Proof. Using (4.14) we have

$$Ca^{-\frac{1}{p'(a)}} \geq \int_{2a}^{4a} x^{-\frac{1}{p'(x)}} \frac{dx}{x} \geq \left(\frac{1}{4a}\right)^{\frac{1}{p'(2a)}} \ln 2 \geq 4^{-\frac{1}{p'(1)}} \ln 2 \left(\frac{1}{a}\right)^{\frac{1}{p'(2a)}};$$

that is,

$$\left(\frac{1}{a}\right)^{\frac{1}{p'(2a)} - \frac{1}{p'(a)}} \leq \frac{4C}{\ln 2}.$$

This proves (4.1) with constant $C_1 = \ln\left(\frac{4C}{\ln 2}\right)$. ■

Lemma 4.4 *Let $p : (0, 1) \rightarrow [1, \infty)$ be a nondecreasing function satisfying the conditions (4.14) and $p(1) < \infty$. Then there exists a constant C_1 such that*

$$\frac{1}{C_1} \phi(x) \leq \phi(y) \leq C_1 \phi(x),$$

for any $\frac{x}{2} < y < 2x$, $0 < x < \frac{1}{4}$, where the function $\phi(t) = t^{-\frac{1}{p'(t)}}$.

Proof. To prove Lemma 4.4 it suffice to apply Lemma 4.3 as in Lemma 4.2. ■

Lemma 4.5 *Let $p : (0, 1) \rightarrow [1, \infty)$ be a nondecreasing function such that $p(1) < \infty$. Then the following two assertions are equivalent:*

- 1) *The condition (4.14) is satisfied.*
- 2) *There exists an $\epsilon > 0$ such that the function $x^\epsilon \phi(x)$ is almost decreasing: there exists a $C_1 > 0$ such that*

$$t_2^\epsilon \phi(t_2) \leq C_1 t_1^\epsilon \phi(t_1), \quad 0 < t_1 \leq t_2 < 1. \quad (4.3)$$

Here the function $\phi(t) = t^{-\frac{1}{p'(t)}}$.

Proof. *Proof of 1) \rightarrow 2).* Denote $g(x) = \int_x^1 \phi(t) \frac{dt}{t}$. Then

$$g'(x) = -\frac{\phi(x)}{x}, \quad 0 < x < 1.$$

Hence

$$g(x) \leq -C g'(x) x \quad \text{or} \quad \frac{1}{C} \frac{1}{x} \leq \frac{-g'(x)}{g(x)}, \quad 0 < x < 1.$$

Integrating this inequality in x over (t_1, t_2) , we get

$$\ln \frac{g(t_1)}{g(t_2)} \geq \frac{1}{C} \ln \frac{t_1}{t_2} \quad \text{or} \quad g(t_2) t_2^{\frac{1}{C}} \leq g(t_1) t_1^{\frac{1}{C}}.$$

Since

$$g(t_2) = \int_{t_2}^1 \phi(x) \frac{dx}{x} \geq \int_{t_2}^{2t_2} \phi(x) \frac{dx}{x} \geq \frac{1}{C} \phi(t_2) \ln 2,$$

using (4.14) and assertion of Lemma 4.4 we get

$$\frac{\ln 2}{C} \phi(t_2) t_2^{\frac{1}{C}} \leq C \phi(t_1) t_1^{\frac{1}{C}}$$

Therefore, (4.3) is satisfied with $\epsilon = \frac{1}{C}$, $C_1 = C^2$.

Proof of 2) \rightarrow 1). Estimating directly, we have

$$\begin{aligned} \int_a^1 \phi(x) \frac{dx}{x} &= \int_a^1 x^\epsilon \phi(x) \frac{dx}{x^{1+\epsilon}} \leq C \int_a^1 a^\epsilon \phi(a) \frac{dx}{x^{1+\epsilon}} \\ &= C a^\epsilon \phi(a) \int_a^1 \frac{dx}{x^{1+\epsilon}} = \frac{C}{\epsilon} \phi(a). \end{aligned}$$

The inequality (4.14) has been proved. ■

Lemma 4.6 *Let $p : (0, 1) \rightarrow [1, \infty)$ be nondecreasing function such that $p(1) < \infty$. Then the condition (4.16) is necessary for the inequality (1.1) to hold.*

Proof. Let $a \in (0, 1)$ be a fixed number. Put a test function

$$f_0(x) = x^{-\frac{1}{p(x)}} \chi_{(\frac{a}{2}, a)}(x), \quad 0 < x < 1,$$

into the inequality (1.1). Then

$$I_{p(\cdot)}(f_0) = \int_{\frac{a}{2}}^a \frac{dx}{x} = \ln 2 \leq 1,$$

therefore, $\|f_0\|_{p(\cdot)} \leq 1$. Hence $\left\| \frac{Hf_0}{x} \right\|_{p(\cdot); (0,1)} \leq C$. This implies that $I_{p(\cdot)}\left(\frac{Hf_0}{x}\right) \leq C_2$, whence

$$\begin{aligned} C_2 &\geq \int_a^1 \left(\int_{\frac{a}{2}}^a t^{-\frac{1}{p(t)}} dt \right)^{p(x)} x^{-p(x)} dx \geq \int_a^1 \left(\frac{a}{2} a^{-\frac{1}{p(a)}} \right)^{p(x)} x^{-p(x)} dx \\ &\geq 2^{-p^+} \int_a^1 \left(a^{\frac{1}{p'(a)}} x^{-\frac{1}{p'(x)}} \right)^{p(x)} \frac{dx}{x}. \end{aligned}$$

Hence

$$\int_a^1 \left(a^{\frac{1}{p'(a)}} x^{-\frac{1}{p'(x)}} \right)^{p(x)} \frac{dx}{x} \leq C_3.$$

■

Lemma 4.7 *Let $p : (0, 1) \rightarrow [1, \infty)$ be a nondecreasing function satisfying the conditions $p(1) < \infty$ and (4.16). Then the function $\phi(x) = x^{-\frac{1}{p'(x)}}$ is almost decreasing; that is for any $0 < t_1 \leq t_2 < 1$ we have*

$$\phi(t_2) \leq C\phi(t_1)$$

Proof. Put $t_1 = a$. Let $2^{k-1}a \leq t_2 < 2^k a$, $k \in \mathbf{N}$. Then using (4.16) and Lemma 4.2, we have

$$\begin{aligned} C &\geq \sum_{n=1}^{\infty} \int_{2^{n-1}a}^{2^n a} \left(a^{\frac{1}{p'(a)}} x^{-\frac{1}{p'(x)}} \right)^{p(x)} \frac{dx}{x} \\ &\geq \sum_{n=1, n \in \mathbf{N}^+}^{\infty} \int_{2^{n-1}a}^{2^n a} \left(a^{\frac{1}{p'(a)}} (2^n a)^{-\frac{1}{p'(2^n a)}} \right)^{p^-} \frac{dx}{x} \\ &\quad + \sum_{n=1, n \in \mathbf{N}''}^{\infty} \int_{2^{n-1}a}^{2^n a} \left(a^{\frac{1}{p'(a)}} (2^n a)^{-\frac{1}{p'(2^n a)}} \right)^{p^+} \frac{dx}{x}, \end{aligned}$$

where $\sum_{n=1, n \in \mathbf{N}^+}^{\infty} (\dots)$ means summing over $n \in \mathbf{N}$ such that $a^{\frac{1}{p'(a)}} (2^n a)^{-\frac{1}{p'(2^n a)}} \geq 1$ and $\sum_{n=1, n \in \mathbf{N}''}^{\infty} (\dots)$ means summing over $n \in \mathbf{N}$ such that $a^{\frac{1}{p'(a)}} (2^n a)^{-\frac{1}{p'(2^n a)}} \leq 1$. Therefore,

$$a^{\frac{1}{p'(a)}} (2^n a)^{-\frac{1}{p'(2^n a)}} \leq 1 + \frac{C}{C_1 \ln 2}, \quad n \in \mathbf{N}. \quad (4.4)$$

Further using the Lemma 4.2, we deduce from (4.4)

$$a^{\frac{1}{p'(a)}} (2^k a)^{-\frac{1}{p'(2^k a)}} \leq C_3,$$

hence by using Lemma 4.2, we have

$$a^{\frac{1}{p'(a)}} t_2^{-\frac{1}{p'(t_2)}} \leq C_4.$$

This completes the proof of Lemma 4.7. \blacksquare

Lemma 4.8 *Let $p : (0, 1) \rightarrow [1, \infty)$ be a nondecreasing function satisfying the conditions $p(1) < \infty$ and (4.16). Then the condition (4.14) is satisfied, moreover, the function $x^{-\frac{1}{p'(x)} + \epsilon}$ is almost decreasing by some $\epsilon > 0$.*

Proof. Using (4.16) and Lemma 4.7 we have the estimates

$$\begin{aligned} C &\geq \int_a^1 \left(a^{\frac{1}{p'(a)}} x^{-\frac{1}{p'(x)}} \right)^{p(x)} \frac{dx}{x} \\ &\geq C_4^{p^-} \int_a^1 \left(\frac{1}{C_4} a^{\frac{1}{p'(a)}} x^{-\frac{1}{p'(x)}} \right)^{p(x)} \frac{dx}{x} \\ &\quad C_4^{p^- - p^+} \int_a^1 \left(a^{\frac{1}{p'(a)}} x^{-\frac{1}{p'(x)}} \right)^{p^+} \frac{dx}{x}. \end{aligned}$$

This implies

$$\int_a^1 x^{-\frac{p^+}{p'(x)}} \frac{dx}{x} \leq C_4^{p(1)-1} a^{-\frac{p^+}{p'(a)}}, \quad 0 < a < 1. \quad (4.5)$$

Applying the approach of Lemmas 4.3 and 4.7, we find the function $x^{-\frac{p^+}{p'(x)}}$ is almost decreasing and satisfies the condition (4.5). It follows from the Bari-Stechkin theorem [1] (see, also [10]) that there exists an $\epsilon > 0$ such that the function $x^{-\frac{p^+}{p'(x)}+\epsilon}$ is almost decreasing. This implies the function $x^{-\frac{1}{p'(x)}+\epsilon_1}$ is almost decreasing. Again using Bari-Stechkin result [1] we deduce the function $x^{-\frac{1}{p'(x)}}$ satisfies the condition (4.14)

Hence we have proved that (by using Lemmas 4.6 and 4.8 for the inequality (1.1) to hold it is necessary the condition (4.14). Let us prove that the condition (4.14) is also sufficient for (1.1). ■

Remark 4.1 *It follows from Lemma 4.8 that the condition (4.14) for nondecreasing $p : (0, 1) \rightarrow [1, \infty)$ implies $p(0) > 1$. Hence the condition $p(0) > 1$ is necessary (but not sufficient) for the inequality (1.1) to hold.*

Lemma 4.9 *Let $p : (0, 1) \rightarrow [1, \infty)$ be a nondecreasing function such that the conditions (4.14) and $p(1) < \infty$ is satisfied. Then the inequality (1.1) holds.*

Proof. Using Lemma 4.3 we infer that the function $x^{-\frac{1}{p'(x)}}$ is almost decreasing. Further, according to Lemma 4.5 the condition (4.14) implies that the function $x^{-\frac{1}{p'(x)}+\epsilon}$ is almost decreasing by some $\epsilon > 0$.

Let us prove sufficiency of condition (4.14). It suffices to consider the case when function $f(x) \geq 0$ is a measurable function such that $\|f\|_{L^{p(\cdot)}(0,1)} \leq 1$ (see, [3]). Then $I_{p(\cdot)}(f) \leq 1$. In order to prove Lemma 4.9 we have to prove $\|x^{-1}Hf\|_{L^{p(\cdot)}(0,1)} \leq C_1$. We shall derive this inequality from the estimate $I_{p(\cdot)}(x^{-1}Hf) \leq C_2$.

By Minkowski inequality, for $L^{p(\cdot)}$ norms, we get the inequalities

$$\begin{aligned} \|x^{-1}Hf\|_{L^{p(\cdot)}(0,1)} &\leq \left\| x^{-\frac{1}{p(x)}-\frac{1}{p(x)}} \sum_{n=0}^{\infty} \int_{2^{-n-1}x}^{2^{-n}x} f(t)dt \right\|_{L^{p(\cdot)}(0,1)} \\ &\leq \sum_{n=0}^{\infty} \left\| x^{-\frac{1}{p(x)}-\frac{1}{p(x)}} \int_{2^{-n-1}x}^{2^{-n}x} f(t)dt \right\|_{L^{p(\cdot)}(0,1)} \end{aligned} \quad (4.6)$$

Denote $B_{x,n} = (2^{-n-1}x, 2^{-n}x]$ and $p_{x,n} = \inf\{p(t) : t \in B_{x,n}\}; n = 1, 2, \dots$. Put $\varphi(t) = t^{\frac{1}{p(t)}}$. Since the condition (4.14) holds, it follows from Lemma 4.8 that there exists an $\epsilon \in (0, 1)$ such that

$$\frac{\varphi(s)}{s^\epsilon} \leq C \frac{\varphi(r)}{r^\epsilon}, \quad 0 < s < r < 1. \quad (4.7)$$

Then by (4.7) we have

$$\frac{\varphi(t)}{t^\epsilon} \leq C \frac{\varphi(x)}{x^\epsilon}, \quad (4.8)$$

where t is a point in $B_{x,n}$, $0 < x < 1$ and the constant C does not depend on n .

By using inequality (4.8) and $2^{-n-1}x < t < 2^{-n}x$ we have the estimates

$$t^{\frac{1}{p'(t)}} = t^\epsilon t^{\frac{1}{p'(t)} - \epsilon} \leq C t^\epsilon x^{\frac{1}{p'(x)} - \epsilon} \leq C 2^{-n\epsilon} x^{\frac{1}{p'(x)}}.$$

Hence

$$x^{-\frac{1}{p'(x)}} \leq C 2^{-n\epsilon} t^{-\frac{1}{p'(t)}}.$$

Therefore, and due to Holder's inequality, for $x \in B(0, 1)$, we get

$$\begin{aligned} & x^{-\frac{1}{p(x)} - \frac{1}{p'(x)}} \sum_{n=0}^{\infty} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \\ & \leq C 2^{-n\epsilon} x^{-\frac{1}{p(x)}} t^{-\frac{1}{p'(t)}} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \\ & \leq C 2^{-n\epsilon} x^{-\frac{1}{p(x)}} t^{-\frac{1}{p'(t)}} \left(\int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}} dt \right)^{\frac{1}{p_{x,n}}} (2^{-n}x)^{\frac{1}{(p_{x,n})'}} \end{aligned} \quad (4.9)$$

It follows from Lemma 4.2 that

$$(2^{-n}x)^{\frac{1}{(p_{x,n})'}} \leq 2^{-\frac{1}{(p_{x,n})'}} t^{\frac{1}{(p_{x,n})'}} \leq C_1 t^{\frac{1}{p'(t)}}, \quad (4.10)$$

where C depends only p .

Combining (4.9) and (4.10) we get

$$x^{-\frac{1}{p(x)} - \frac{1}{p'(x)}} \sum_{n=0}^{\infty} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \leq C 2^{-n\epsilon} x^{-\frac{1}{p(x)}} \left(\int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}} dt \right)^{\frac{1}{p_{x,n}}} \quad (4.11)$$

where $0 < x < 1$, $n = 1, 2, \dots$ and the constant C_2 does not depend on n, x .

Simultaneously,

$$\int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}} dt \leq \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p(t)} \chi_{\{f(t) \geq 1\}} dt + \int_{2^{-n-1}x}^{2^{-n}x} dt \leq 1 + 2^{-n} \leq C_3.$$

By the last inequality and (4.11), we have

$$\begin{aligned} I_{p(\cdot)} \left(x^{-\frac{1}{p(x)} - \frac{1}{p'(x)}} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \right) & \leq C_4 2^{-n\epsilon p^-} \int_0^1 x^{-1} \left(\int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}} dt \right)^{\frac{p(x)}{p_{x,n}}} dx \\ & \leq C_4 C_3^{\frac{p^+}{p^-} - 1} 2^{-n\epsilon p^-} \int_0^1 x^{-1} \left(\int_{2^{-n-1}x}^{2^{-n}x} (f(t)^{p(t)} + 1) dt \right) dx \end{aligned}$$

which, due to Fubini's theorem, yields

$$\begin{aligned}
&\leq C_4 C_3^{\frac{p^+}{p^-}-1} 2^{-n\epsilon p^-} \ln 2 \int_0^{2^{-n}} \left(\int_{2^{-n-1}x}^{2^{-n}x} x^{-1} dx \right) \left(f(t)^{p(t)} + 1 \right) dt \\
&= C_5 2^{-n\epsilon p^-} \ln 2 \int_0^{2^{-n}} \left(f(t)^{p(t)} + 1 \right) dt \leq C_6 2^{-n\epsilon p^-}. \tag{4.12}
\end{aligned}$$

Therefore,

$$\left\| x^{-\frac{1}{p(x)} - \frac{1}{p'(x)}} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \right\|_{L^{p(\cdot)}(0,1)} \leq C 2^{-\frac{n\epsilon p^-}{p^+}}$$

By (4.12) and (4.6), we get

$$\|x^{-1} Hf\|_{L^{p(\cdot)}(0,1)} \leq C \sum_{n=0}^{\infty} 2^{-\frac{n\epsilon p^-}{p^+}} \leq C_1.$$

This completes the proof of Lemma 4.9. \blacksquare

Proof of Theorem 2.2. Let 5) be satisfied, that is the condition (4.17). Then by the definition,

$$\int_a^1 \left(\frac{x^{-1}}{\|(\cdot)^{-1} \chi_{\{a,1\}}(\cdot)\|_{p(\cdot)}} \right)^{p(x)} dx \leq 1.$$

Therefore, and using (4.17), we have

$$\int_a^1 \left(\frac{x^{-1}}{C a^{-\frac{1}{p'(a)}}} \right)^{p(x)} dx \leq 1$$

or

$$\int_a^1 \left(a^{\frac{1}{p'(a)}} x^{-\frac{1}{p'(x)}} \right)^{p(x)} \frac{dx}{x} \leq C_1.$$

This is the condition (4.16), that is 4) of Theorem 2.2. Hence 5) \rightarrow 4) has been proved. According to Lemmas 4.6, 4.7, 4.8, we have the implication 4) \rightarrow 2). The implication 2) \rightarrow 3) follows from Lemma 4.5. The implication 3) \rightarrow 1) follows from Lemma 4.9. The implication 1) \rightarrow 4) is proved in Lemma 4.6.

The implication 3) \rightarrow 5) is direct: using the condition (4.15) we have

$$\begin{aligned}
&\int_a^1 \left(a^{\frac{1}{p'(a)}} x^{-\frac{1}{p'(x)}} \right)^{p(x)} \frac{dx}{x} \leq \int_a^1 \left(C \left(\frac{a}{x} \right)^\epsilon \right)^{p(x)} \frac{dx}{x} \\
&= \int_1^{\frac{1}{a}} C^{p(1)} \left(\frac{1}{t} \right)^{\epsilon p(at)} \frac{dt}{t} \leq C^{p(1)} \int_1^\infty \frac{dt}{t^{1+\epsilon p(0)}} < C_2.
\end{aligned}$$

Rewriting the last inequality, we have

$$\int_a^1 \left(\frac{x^{-1}}{C_2^{\frac{1}{p(1)}} a^{-\frac{1}{p'(a)}}} \right)^{p(x)} dx \leq 1,$$

therefore, the condition (4.17) is satisfied.

This completes the proof of Theorem 2.2.

If the exponent function p in Theorem 2.2 is nondecreasing on not all the interval $(0, 1)$ but so is only near the origin the following assertion holds.

Remark 4.2 *Let a measurable function $p : [0, 1] \rightarrow (1, \infty)$ be nondecreasing on some interval $(0, \delta)$, $0 < \delta < 1$ and $p^+ < \infty$; then the following statements are equivalent:*

a) *There exists a constant $C > 0$ such that the inequality*

$$\|x^{-1} Hf\|_{L^{p(\cdot)}(0,1)} \leq C \|f\|_{L^{p(\cdot)}(0,1)} \quad (4.13)$$

holds for any positive measurable function f .

b) *The condition*

$$\int_a^\delta x^{-\frac{1}{p'(x)}} \frac{dx}{x} \leq C a^{-\frac{1}{p'(a)}}, \quad 0 < a < \delta \quad (4.14)$$

is satisfied.

c) *There exists an $\epsilon > 0$ such that the function $x^{-\frac{1}{p'(x)} + \epsilon}$ is almost decreasing:*

$$t_2^{-\frac{1}{p'(t_2)} + \epsilon} \leq C t_1^{-\frac{1}{p'(t_1)} + \epsilon} \quad \text{as} \quad 0 < t_1 \leq t_2 < \delta \quad (4.15)$$

d) *The condition*

$$\int_a^\delta \left(a^{\frac{1}{p'(a)}} x^{-\frac{1}{p'(x)}} \right)^{p(x)} \frac{dx}{x} \leq C, \quad 0 < a < \delta \quad (4.16)$$

is satisfied.

e) *The condition*

$$\|x^{-1}\|_{p(\cdot);(a,\delta)} \leq C a^{-\frac{1}{p'(a)}}, \quad 0 < a < \delta. \quad (4.17)$$

is satisfied.

References

- [1] N.K. Bari and S.B. Stechkin. "Best approximations and differential properties of two conjugate functions" (in Russian). *Proceedings of Moscow Mathematical Society*, 5:483-522, 1956.
- [2] D. Cruz-Uribe, SFO and F. I. Mamedov, "On a general weighted Hardy type inequality in the variable exponent Lebesgue spaces," *Revista Matemática Complutense*, vol. 25, no. 2, pp. 335-367, 2012.
- [3] L. Diening, P. Harjulehto, P. Hasto and M. Ruzicka, "Lebesgue and Sobolev Spaces with Variable Exponents," *Lecture Notes in Mathematics*, vol 2017, Springer, Heidelberg, Germany, 2011.
- [4] L. Diening and S. Samko, "Hardy inequality in variable exponent Lebesgue spaces," *Fractional Calculus & Applied Analysis*, vol. 10, no 1, pp. 1-17, 2007
- [5] D. E. Edmunds, V. Kokilashvili and A. Meskhi, "On the boundedness and compactness of the weighted Hardy operators in spaces," *Georgian Mathematical Journal*, vol. 12, no. 1, pp. 27-44, 2005.
- [6] X. L. Fan and D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 424-446, 2001.
- [7] A. Harman, "On necessary condition for the variable exponent Hardy inequality," *Journal of Function Spaces and Applications*, vol. 2012, Article ID 385925, 6 pages, doi:10.1155/2012/385925
- [8] P. Harjulehto, P. Hasto and M. Koskinoja, "Hardy's inequality in variable exponent Sobolev spaces," *Georgian Mathematical Journal*, vol. 12, no. 3, pp. 431-442, 2005.
- [9] A. Harman and F.I. Mamedov, "On boundedness of weighted Hardy operator in $L^{p(\cdot)}$ and regularity condition," *Journal of Inequalities and Applications*, vol. 2010, Article ID 837951, 14 pages, 2010.
- [10] V. Kokilashvili, S. Samko and N. Samko, "The Maximal Operator in Weighted Variable Spaces $L^{p(\cdot)}$ " *J. Function Spaces Appl.*, vol. 5, no 3, pp. 299-317, 2007.
- [11] F. I. Mamedov, "On Hardy type inequality in variable exponent Lebesgue space $L^{p(\cdot)}(0, 1)$," *Azerbaijan Journal of Mathematics*, vol. 2, no. 1, pp. 90-99, 2012.
- [12] F.I. Mamedov and A. Harman, "On a weighted inequality of Hardy type in spaces $L^{p(\cdot)}$," *Journal of Mathematical Analysis and Applications*, vol. 353, no. 2, pp. 521-530, 2009.

- [13] F.I. Mamedov and A. Harman, "On a Hardy type general weighted inequality in spaces $L^{p(\cdot)}$," *Integral Equations and Operator Theory*, vol. 66, no. 4, pp. 565-592, 2010.
- [14] F.I. Mamedov and Y. Zeren, "On equivalent conditions for the general weighted Hardy type inequality in space $L^{p(\cdot)}$," *Zeitschrift fur Analysis und ihre Anwendungen*, vol. 31, no 1, pp. 55-74, 2012
- [15] R. Mashiyev, B. Cekic, F. I. Mamedov and S. Ogrash, "Hardy's inequality in power-type weighted $L^{p(\cdot)}$ spaces," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 1, pp. 289-298, 2007.
- [16] H. Rafeiro and S. G. Samko, "Hardy inequality in variable Lebesgue spaces," *Annales Academiae Scientiarum Fennicae*, vol. 34, no. 1, pp. 279-289, 2009.
- [17] S. G. Samko, "Hardy inequality in the generalized Lebesgue spaces," *Fractional Calculus & Applied Analysis*, vol. 6, no. 4, pp. 355-362, 2003.
- [18] S. G. Samko, "Convolution type operators in $L^{p(\cdot)}$," *Integral Transforms and Special Functions*, vol. 7, pp. 123-144, 1998.